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A REDUCED-ORDER ANALYTICAL MODEL FOR THE NONLINEAR DYNAMICS OF A CLASS OF FLEXIBLE MULTI-BEAM STRUCTURES

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Abstract—Nonlinearities in the differential equations of motion of dynamical systems can play, under certain conditions, such a dominant role that the motion described by linearized differential equations bears no resemblance to the actual motion exhibited by the systems. For a structure, nonlinearities are due to material behavior and to deformation. The latter are called "geometric nonlinearities" and, even for linear (i.e., Hookean) materials and small deformations, their effect can be dramatic. To investigate the nonlinear behavior of a dynamical system by making use of analytical techniques, one must start the analysis by formulating a set of mathematically consistent differential equations of motion for the system. Furthermore, the equations must be cast in a form that makes them amenable to the application of known analytical methods, such as perturbation techniques, to investigate the motion. The work presented in this paper addresses the formulation of such equations for a class of multi-beam structures. Each beam in the structure may have arbitrary cross section variation along its span, but behaves as inextensional. The structure may have any number of beams and supports, and may carry any number of lumped masses along its span.

INTRODUCTION AND LITERATURE REVIEW

The differential equations of motion for dynamical systems, such as mechanical and structural systems, are inherently nonlinear. It is well known that linearized differential equations about the equilibrium solutions of dynamical systems (which are obtained from a solution of nonlinear equations) may or may not yield a valid approximation to the actual response of the system, even for very small motions about the equilibrium (see, for example, Haight and King, 1972; Ho *et al.*, 1975, 1976; Crespo da Silva and Glynn, 1978a, 1978b). A nonlinear analysis of the dynamics of such systems is of crucial importance of predicting and fully understanding their behavior under the effect of applied loads. The first step in an analysis based on nonlinear models is, generally, a first-order linearized analysis of small perturbed motions of the system about its stable equilibrium solutions. To determine the effect of the nonlinearities, one may proceed with a higher-order perturbation analysis of the motion. Such perturbation analyses are able to disclose the situations when the nonlinearities become important, and to yield important details about the system's response that are either impractical or nearly impossible to obtain with purely numerical simulation methodologies. A thorough presentation of a number of perturbation methods, with application to several discrete and continuous systems, is found in Nayfeh and Mook (1979).

The study of nonlinear dynamics of flexible systems has been the subject of a number off investigations presented in the literature. Both numerical and analytical investigations have been presented by a number of authors.

Numerical investigations have the advantage of being able to deal with large systems. However, numerical investigations of the nonlinear dynamics of flexible structural systems present a number of well known problems. Such investigations have generally been confined to obtaining numerical solutions to equations of motion with the objective of simulating the time response of the system due to applied loads. Although results obtained from such numerical simulations are certainly very useful for design purposes, they are generally difficult (if not impossible) to interpret and to generalize in the presence of nonlinearities due to the many possible choices for the system parameters and initial conditions for the motion. Some advance knowledge of the dynamical behavior of the system may even be needed in order to avoid a partial, or even erroneous, interpretation of purely numerical results. For example, analytical investigations have shown that different types of nonlinearities (namely, inertia and curvature nonlinearities) in the differential equations of motion of a beam may act in such a way that their effects could nearly cancel themselves for some range of excitation frequencies and system parameters (Crespo da Silva and Glynn, 1978a, 1978b). By relying only on results of numerical simulations for such cases, the investigator might reach the erroneous conclusion that the system is essentially linear while, in fact, it is not. This was first shown in Crespo da Silva and Glynn (1978b) for a single beam. As disclosed in that reference, the effects of the same nonlinearities for different values of excitation frequencies and system parameters can make the results of a linear analysis totally incorrect, even for motions that might be labeled "small".

Analytical techniques such as perturbation methods play a significant role in the analysis of the response of structural systems. They provide a great deal of insight into the physics of a given problem and they allow the analyst to separate and handle the effects of different nonlinearities in the system. However, they have been applied only to dynamical systems with a relatively small number of degrees of freedom. For flexible systems, they have been applied to simpler systems such as a single structural element, but not to complex systems made of a number of interconnected elements.

The problem of dealing with structural systems that are composed of a number of interconnected structural elements consists in the high dimensionality of such systems. This has made them intractable with purely analytical methods. It is perhaps for this reason that the work that addresses such systems has been confined to numerical investigations. Much of that work deals with the development of numerical methods to obtain the static and/or the dynamic response of the system.

A significant amount of work has been done by a number of investigators dealing with numerical methods for the determination of the static deflection and the numerical simulation of the dynamics of flexible structures. Such numerical work is exemplified in Hsiao *et al.* (1979), Surana and Sorem (1989), Bathe and Bolourchi (1979), Belytschko *et al.* (1977), Simo (1985), Simo and Vu-Quoc (1991), Cardona and Géradin (1988), Iura and Atluri (1988), Park *et al.* (1991), Downer *et al.* (1992), and in the many references cited in those papers. Reduction methods have also been used to improve the efficiency of generating numerical solutions to problems involving flexible structures, both for a single member and for more complex structures. With such methods, the structure is approximated by a system with a reduced number of properly selected degrees of freedom by appropriate choice of motion coordinates. Such work is exemplified in Noor (1981, 1982, 1985), and in Noor and Peters (1980a, 1980b). In the work presented in Noor (1982), the author also briefly discusses the selection of the motion coordinates for structural dynamics problems with the objective of numerically solving the finite element equations of motion in a more efficient manner in order to obtain the response of the system as a function of time.

The analytical work that is found in the literature dealing with the dynamic response of structural systems consists in investigations of the dynamic response of the system when it is perturbed about its equilibrium position, including investigations that involve the effects of nonlinearities in the motion of the system. Clearly, one needs to have an explicit set of differential equations of motion to conduct such investigations. To perform such investigations using perturbation techniques, the differential equation of motion need to be cast in a form that contain polynomial nonlinearities in the motion variables when such variables are perturbed about a particular solution (such as an equilibrium state) of the system.

OBJECTIVE OF THE PRESENT WORK

The objective of this paper is to develop an *explicit* set of reduced-order nonlinear differential equations of motion for a class of multi-beam structures. The equations are put in a form suited for the use of analytical techniques for predicting nonlinear phenomena and investigating nonlinear motions exhibited by the structure when its equilibrium state is perturbed. Analytical investigations based on such equations can yield a wealth of information about the physics of the problem. The equations developed here are also suited for designing nonlinear control systems for the structure taking into account the geometric nonlinearities that appear due to bending of any of its members.

Of special relevance to the work presented here are the works in Crespo da Silva (1997) and in Crespo da Silva and Glynn (1978a, 1978b).

In Crespo da Silva and Glynn (1978a), an explicit set of fully nonlinear partial differential equations for inextensional beams were developed by taking into account all the geometric nonlinearities (due to curvature and inertia) that occur when the beam deforms. Those equations were then expanded about the equilibrium state of the beam for the case when that state coincides with the undeformed state of the system. The expanded differential equations developed in Crespo da Silva and Glynn (1978a) were reduced to nonlinear integro-partial differential equations in Crespo da Silva and Glynn (1978b), and the latter equations were then used to investigate analytically the nonlinear resonant response of the system to an external excitation. This was done with a perturbation method where the eigenfunctions obtained from the linearized differential equations were used in a modal reduction Galerkin's technique to approximate the higher order partial differential equations by a finite set of ordinary differential equations. Similar studies were presented in Crespo da Silva (1988a, 1988b) for beams with fixed supports, which are extensional beams.

In Crespo da Silva (1997), a set of reduced-order nonlinear ordinary differential equations for inextensional beams were obtained directly by applying the modal reduction technique to the expression for Hamilton's principle. Furthermore, the work in Crespo da Silva (1997) allowed for arbitrary equilibrium deflection of the beam due to the application of static loads and to the existence of any number of concentrated masses along the beam's span. As in Galerkin's procedure, the spatial functions used in the modal reduction technique in Crespo da Silva (1997) were chosen to be the eigenfunctions associated with the linearized set of partial differential equations that govern the motion of the infinitesimally small perturbations about the equilibrium state of the system. The ordinary differential equations obtained by such a direct procedure are the same as those obtained by applying Galerkin's procedure to the nonlinear partial differential equations referred to above, although the expressions for the coefficients of similar terms that appear in the two sets of equations may look different from each other. However, an integration by parts in the coefficients in any of the two sets of equations discloses the equivalency of the two sets of equations.

The formulation presented in Crespo da Silva (1997) is extended here to a particular class of multi-beam structures in planar motion. The structures under consideration may be made of an arbitrary number, N_{beams}, of Hookean beams, connected in such a way that each beam in the system can be approximated as inextensional. For such structures, the axial component of the deformation of each of its members can be readily expressed in terms of the transverse component (i.e., perpendicular to the undeformed direction of the member) of the elastic deformation by making use of the inextensionality condition for that member. Each beam in the structure is assumed to be straight, when undeformed, and it may have variable cross section along its span (which imply arbitrary mass and stiffness distribution). The connections between any two beams may consist of pins, welds, or sliding joints. In addition, the structure may also carry any number N_{conc} of concentrated masses placed anywhere on the system. The reduced-order nonlinear differential equations of motion formulated here account for nonlinear effects due to elastic deformation of its members.

DEVELOPMENT OF THE REDUCED-ORDER MODEL

To develop a set of reduced-order nonlinear differential equations that approximate the dynamic behavior of the entire structure, we start by dividing each beam in the structure into a desired number of "elements". Each one of these elements will be referred to in the

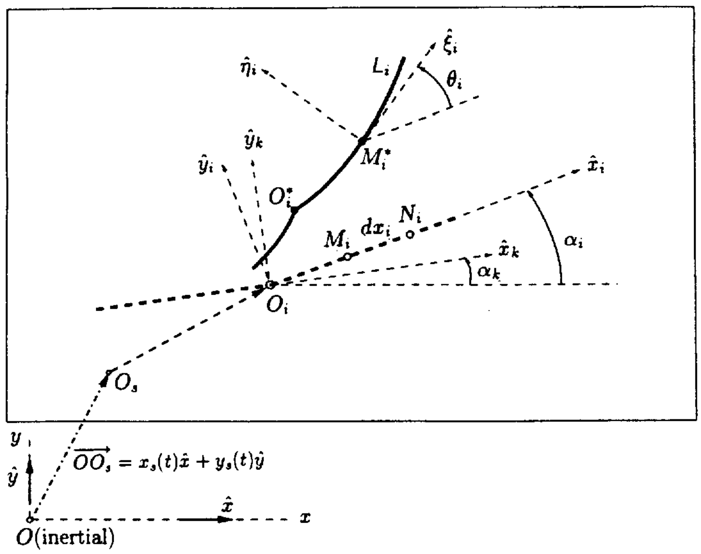

Fig. 1. A beam element connected to another element.

sequel as a *beam element*, and the total number of beam elements in the structure is N_{elem}. To begin with, let us then concentrate on the dynamics of an arbitrary beam element.

Figure 1 shows an arbitrary thin beam element i of the structure, before and after deformation, connected to another beam element k at point O_i which, by definition, is at the "left boundary" of element i. Before deformation, each beam element is straight and untwisted. Each beam element i (for $i = 1, 2, \ldots, N_{\text{elem}}$) has length L_i meters, distributed mass density $m_i(s_i)$ Kg/meter and bending stiffness $D_i = E_iI_i(s_i)$ Newton.meter^2, where E_i and I_i are, respectively, the modulus of elasticity and the appropriate area moment of inertia for the element's cross section (which will be referred to by $\mathbf{S}_i$). The cross section $\mathbf{S}_i$ along the element, with area A_i, may be variable but the beam element is assumed to be made of a Hookean material. The independent variable s_i denotes arc length measured along the deformed reference line of beam element i (which is line $\mathbf{L}_i$ shown in Fig. 1).

The quantities with a caret (such as $\hat{x}_i$, $\hat{x}_k$, etc.) shown in Fig. 1 denote unit vectors. The reference axes x and y shown in Fig. 1 maintain a constant direction in inertial space; $\hat{x}$ and $\hat{y}$ are unit vectors along those directions. The x_i and y_i axes (with unit vectors $\hat{x}_i$ and $\hat{y}_i$, respectively) are aligned with beam element i before deformation, with x_i inclined at an angle α_i with the reference x-axis. If gravity is neglected (i.e., if the acceleration of gravity, $\mathbf{g}$, is set to zero by the user of the differential equations that will be developed in this paper), the spatial orientation of the xy plane is arbitrary. Otherwise, the y reference line is taken to be vertical, with the gravitational forces being directed along $-\hat{y}$. The $\hat{\xi}_i$ and $\hat{\eta}_i$ unit vectors are aligned with the principal directions of the element cross section after deformation. Before deformation, beam element i is a straight line that is aligned with x_i, and the length of an infinitesimal segment M_iN_i along x_i is $\mathrm{d}x_i$. After deformation, M_i is located at M_i^*, and the length of the same segment is $\mathrm{d}s_i$; the angle θ_i shown in Fig. 1 is the angle between $\hat{x}_i$ and $\hat{\xi}_i$ at M_i^* after deformation.

The vector $\overrightarrow{OO_s} = x_s(t)\hat{x} + y_s(t)\hat{y}$ (see Fig. 1), from the inertial point O to any point O_s fixed to the plane of motion of the structure (shown as a rectangle in Fig. 1), accounts for a prescribed motion that plane may be forced to undergo in any direction parallel to the x–y plane. The quantities $x_s(t)$ and $y_s(t)$, where t denotes time, are the x and y inertial components of the external "base excitation" the entire structure may be subjected to.

The elastic deformation vector of a point M_i on the reference line x_i of beam element i can be written as $u_i(s_i, t)\hat{x}_i + v_i(s_i, t)\hat{y}_i$, where $v_i(s_i, t)$ is the elastic displacement of M_i due to bending. Let $e_{0i} = \partial s_i/\partial x_i - 1 = \sqrt{(1 + \partial u_i/\partial x_i)^2 + (\partial v_i/\partial x_i)^2} - 1$. For inextensional beam elements, $e_{0i} = 0$. Here, the small effect of shear will be neglected, and the orientation of the cross section $\mathbf{S}_i$ is perpendicular to $\mathbf{L}_i$ at M_i^* during deformation. After deformation, the absolute position vector of an arbitrary point P_i^* in the cross section $\boldsymbol{S}_i$ is then expressed as shown in eqn (1) below, where $\hat{z}$ is a unit vector perpendicular to the x–y plane and is equal to the cross product $\hat{x} \times \hat{y}$.

$$\vec{r}_i = \overrightarrow{OP_i^*} = \underbrace{\overrightarrow{OO_s}}_{x_s(t)\hat{x}+y_s(t)\hat{y}} + \underbrace{\overrightarrow{O_sO_i}}_{\text{a constant}} + 1131\underbrace{\overrightarrow{O_iO_i^*}}_{(x_i+u_i)\hat{x}_i+v_i\hat{y}_i}1131 + \overrightarrow{O_i^*M_i^*} + \underbrace{\overrightarrow{M_i^*P_i^*}}_{\eta_i(\hat{y}_i\cos\theta_i - \hat{x}_i\sin\theta_i)+\zeta_i\hat{z}} \tag{1}$$

In the above equation, η_i and $\zeta_i = z_i$ are the local coordinates of point P_i^* relative to M_i^*. As indicated above, the vectors $\overrightarrow{OO_s}$ and $\overrightarrow{O_sO_i}$ are known quantities.

As mentioned earlier, the work presented here will be restricted to the class of structures where the behavior of each beam in the system can be approximated as inextensional (i.e., its length, after deformation, is equal to its undeformed length). For such problems, the following constraint conditions holds for each beam element, where primes denote partial differentiation with respect to s_i.

$$(1+u_i')^2 + v_i'^2 = 1 \tag{2}$$

Both $\theta_i(s_i, t)$ and $u_i(s_i, t)$ can be eliminated from the formulation by expressing them in terms of the bending deflection $v_i(s_i, t)$. The constraint condition, eqn (2), yields the expression for u_i' in terms of v_i', while $\theta_i(s_i, t)$ can be expressed in terms of $v_i'(s_i, t)$ by noticing that $\sin\theta_i(s_i, t) = v_i'(s_i, t)$.

To investigate the dynamic response of the structure about its equilibrium state, the bending deflection $v_i(s_i, t)$ for each element of the structure is first expressed as

$$v_i(s_i, t) = v_{ie}(s_i) + v_{is}(s_i, t) \tag{3}$$

where $v_{ie}(s_i)$ is the static equilibrium bending deflection, and $v_{is}(s_i, t)$ is a perturbation about the equilibrium. To clarify the notation adopted here, a subscript e will always be used to denote the equilibrium value of a variable, such as $v_{ie}(s_i)$ (which stands for the equilibrium v-deflection along element i). A subscript s, in turn, is used to indicate a perturbed state of a variable about its equilibrium (such as $v_{is}(s_i, t)$, which stands for the perturbed v-deflection along element i).

By solving eqn (2) for $u_i'(s_i, t)$ in terms of $v_i'(s_i, t) = v_{ie}'(s_i) + v_{is}'(s_i, t)$, expanding the solution in power series in the perturbation deflection, truncating the series to the fourth degree in the perturbation, and by noticing that $\sin\theta_{is}(s_i) = v_{is}'(s_i)$ and $\cos\theta_{ie}(s_i) = 1 + u_{ie}'(s_i)$, the following expression is obtained for $u_i'(s_i, t)$. The symbol $\triangleq$ denotes "equal to by definition".

$$\begin{aligned} u_i'(s_i, t) &= u_{ie}'(s_i) - \left[(\tan\theta_{ie}) v_{is}' + \frac{v_{is}'^2}{2\cos^3\theta_{ie}} + \frac{(\sin\theta_{ie}) v_{is}'^3}{2\cos^5\theta_{ie}} + \frac{(1+4\sin^2\theta_{ie}) v_{is}'^4}{8\cos^7\theta_{ie}} \right] + \cdots \\ &\triangleq u_{ie}'(s_i) + u_{is}'(s_i, t) \end{aligned} \tag{4}$$

In addition, the following expression for $\theta_i(s_i, t)$ is also obtained by expanding $\theta_i(s_i, t) = \arcsin[v_{ie}'(s_i) + v_{is}'(s_i, t)]$.

$$\begin{aligned} \theta_i(s_i, t) &= \theta_{ie}(s_i) + \frac{1}{\cos\theta_{ie}} v_{is}' + \frac{\sin\theta_{ie}}{2\cos^3\theta_{ie}} v_{is}'^2 \\ &\quad + \frac{1}{\cos^3\theta_{ie}} \left[\frac{1}{6} + \frac{1}{2}\tan^2\theta_{ie} \right] v_{is}'^3 + \frac{\sin\theta_{ie}}{8\cos^5\theta_{ie}} [3 + 5\tan^2\theta_{ie}] v_{is}'^4 + \cdots \\ &\triangleq \theta_{ie}(s_i) + A_{1i}(\theta_{ie}) v_{is}' + A_{2i}(\theta_{ie}) v_{is}'^2 + A_{3i}(\theta_{ie}) v_{is}'^3 + A_{4i}(\theta_{ie}) v_{is}'^4 + \cdots \end{aligned} \tag{5}$$

The differential equations that govern the motion of any point M_i on an arbitrary beam element i of the structure can be obtained either by a vectorial approach using Newton's second law, or by a variational formulation. The use of either is a matter of individual preference. Because of its relative simplicity in handling even the most complex systems and, at the same time, because it eliminates, at the beginning of the formulation, all the constraint forces (which are forces whose virtual work is zero), the variational formulation will be used. Here, as in Crespo da Silva and Glynn (1978a) and in Crespo da Silva (1991), use will be made of Hamilton's extended principle (e.g., Lanczos, 1966; Pars 1975; Washizu, 1975; Meirovitch, 1970) to generate the differential equations of motion for the system. For this, the expressions for the kinetic energy of the motion, and for the virtual work done by the forces acting on the system are needed.

By making use of eqn (1) for the position vector for an arbitrary point of an element of the structure, the specific (i.e., per unit length) kinetic energy, T_i, associated with the motion of any element i of the structure is then obtained as shown in the following expression, where $J_{\zeta_i} = \int_{S_i} \eta_i^2 \, dm_i$ and overdots denote partial differentiation with respect to time.

$$\begin{aligned} T_{i_{\text{dist}}} &= \frac{1}{2}\int_{S_i} \frac{d\vec{r}_i}{dt}\cdot\frac{d\vec{r}_i}{dt}\,dm_i \\ &= \frac{1}{2}m_i[\dot{x}_s^2+\dot{y}_s^2+\dot{u}_i^2+\dot{v}_i^2+2(\dot{x}_s\dot{u}_i+\dot{y}_s\dot{v}_i)\cos\alpha_i+2(\dot{y}_s\dot{u}_i-\dot{x}_s\dot{v}_i)\sin\alpha_i] \\ &\quad+\frac{1}{2}J_{\zeta_i}\dot{\theta}_i^2-\dot{\theta}_i[\dot{u}_i\cos\theta_i+\dot{v}_i\sin\theta_i+\dot{x}_s\cos(\alpha_i+\theta_i)+\dot{y}_s\sin(\alpha_i+\theta_i)]\int_{S_i}\eta_i\,dm_i \end{aligned} \tag{6}$$

The integral that appears in the last term in eqn (6) is zero if the reference line L_i is chosen to be the line that passes through the mass centroids of the consecutive cross sections S_i. To simplify the expression for the specific kinetic energy, such choice will then be made from now on. The term proportional to $\dot{\theta}_i^2$ in eqn (6) is recognized as the kinetic energy due to rotation of the cross section S_i, while the terms that are independent of $\dot{\theta}_i$ constitute the kinetic energy due to translation of the reference point M_i. Since the distributed mass moment of inertia of a thin beam is a very small quantity, the kinetic energy due to rotation of the cross section is several orders of magnitude smaller than the kinetic energy due to translation of M_i and, thus, will be neglected. Such approximation is also consistent with neglecting the effect of shear.

Let the structure also carry an arbitrary number N_{conc} of concentrated masses along its span, each of mass $M_{i_{\text{conc}}}$, and moment of inertia $J_{i_{\text{conc}}}$ (where $i_{\text{conc}} = 1, 2, \ldots, N_{\text{conc}}$). The expression for the kinetic energy, $T_{i_{\text{conc}}}$, associated with the motion of an arbitrary concentrated mass $M_{i_{\text{conc}}}$ is then

$$\begin{aligned} T_{i_{\text{conc}}} &= \frac{1}{2}M_{i_{\text{conc}}}[\dot{x}_s^2+\dot{y}_s^2+\dot{u}_{i_{\text{conc}}}^2+\dot{v}_{i_{\text{conc}}}^2+2(\dot{x}_s\dot{u}_{i_{\text{conc}}}+\dot{y}_s\dot{v}_{i_{\text{conc}}})\cos\alpha_{i_{\text{conc}}} \\ &\quad+2(\dot{y}_s\dot{u}_{i_{\text{conc}}}-\dot{x}_s\dot{v}_{i_{\text{conc}}})\sin\alpha_{i_{\text{conc}}}]+\frac{1}{2}J_{i_{\text{conc}}}\dot{\theta}_{i_{\text{conc}}}^2 \\ &\quad-\dot{\theta}_{i_{\text{conc}}}[\dot{u}_{i_{\text{conc}}}\cos\theta_{i_{\text{conc}}}+\dot{v}_{i_{\text{conc}}}\sin\theta_{i_{\text{conc}}}+\dot{x}_s\cos(\alpha_{i_{\text{conc}}}+\theta_{i_{\text{conc}}}) \\ &\quad+\dot{y}_s\sin(\alpha_{i_{\text{conc}}}+\theta_{i_{\text{conc}}})]\int_{M_{i_{\text{conc}}}}\eta_{i_{\text{conc}}}\,dM_{i_{\text{conc}}} \end{aligned} \tag{7}$$

The integral that appears in eqn (7), is equal to $M_{i_{\text{conc}}}$ times the offset distance between the reference axis of the element to which mass $M_{i_{\text{conc}}}$ is attached, and the center of mass of $M_{i_{\text{conc}}}$. For simplicity, this offset distance will be set to zero from now on (which implies that the mass centroid of each lumped mass is at the reference line L_i).

For the derivation of the expression for the specific strain energy, U_i, for the general case that involves bending and extension, and when the element's reference line L_i does not pass through the area centroid of the cross section S_i, the reader is referred to Crespo da Silva (1988a). If L_i for element i is chosen to pass through the area centroid of S_i (which is a simplifying assumption that will be adopted from now on, thus implying that the mass and area centroids of S_i coincide with each other), and if the beam's material is Hookean, the expression for the specific strain energy for an inextensional element is simply

$$U_i = \tfrac{1}{2} D_i \theta_i'^2 \tag{8}$$

where $D_i = E_i I_i = E_i \int_{S_i} \eta_i^2 \, dA_i$ is the element's bending stiffness.

The specific gravitational potential energy for element i, $U_{i_{\text{grav}}}$, is (except for a constant) given by eqn (9) shown below. The quantity g is the acceleration of gravity. The potential energy associated with a concentrated mass $M_{i_{\text{conc}}}$ is of the same form as eqn (9), with the subscript i replaced by i_{conc} and with m_i replaced by $M_{i_{\text{conc}}}$.

$$U_{i_{\text{grav}}} = m_i g \vec{r}_i \cdot \hat{y} = m_i g [y_s(t) + (x_i + u_i) \sin \alpha_i + v_i \cos \alpha_i] \tag{9}$$

As indicated before, the differential equations of motion for each beam element will be developed using Hamilton's variational principle, which can be written as eqn (10) below (where $s_{i_{\text{conc}}}$ denotes the value of s_i, along beam element i, where $M_{i_{\text{conc}}}$ is located).

$$\begin{aligned}
\delta I \triangleq \delta \int_{t=t_1}^{t_2} \sum_{i=1}^{N_{\text{elem}}} \int_0^{L_i} \Big\{ & \frac{1}{2} m_i [\dot{x}_s^2 + \dot{y}_s^2 + \dot{u}_i^2 + \dot{v}_i^2 \\
& + 2(\dot{x}_s \dot{u}_i + \dot{y}_s \dot{v}_i) \cos \alpha_i + 2(\dot{y}_s \dot{u}_i - \dot{x}_s \dot{v}_i) \sin \alpha_i] - D_i \theta_i'^2 \Big\} \, ds_i \, dt \\
+ \delta \int_{t=t_1}^{t_2} \Big\{ \sum_{i=1}^{N_{\text{conc}}} \Big[& \frac{1}{2} M_{i_{\text{conc}}} [\dot{x}_s^2 + \dot{y}_s^2 + \dot{u}_i^2 + \dot{v}_i^2 \\
& + 2(\dot{x}_s \dot{u}_i + \dot{y}_s \dot{v}_i) \cos \alpha_i + 2(\dot{y}_s \dot{u}_i - \dot{x}_s \dot{v}_i) \sin \alpha_i] + \frac{1}{2} J_{i_{\text{conc}}} \dot{\theta}_i^2 \Big]_{s_i = s_{i_{\text{conc}}}} \Big\} dt \\
- \int_{t=t_1}^{t_2} \int_0^{L_i} & \sum_{i=1}^{N_{\text{elem}}} m_i g [(\cos \alpha_i) \delta v_i + (\sin \alpha_i) \delta u_i] \, ds_i \, dt \\
- \int_{t=t_1}^{t_2} \Big\{ & \sum_{i=1}^{N_{\text{conc}}} \Big[M_{i_{\text{conc}}} g [(\cos \alpha_i) \delta v_i + (\sin \alpha_i) \delta u_i] \Big]_{s = s_{i_{\text{conc}}}} \Big\} dt + \int_{t=t_1}^{t_2} \delta W \, dt = 0
\end{aligned} \tag{10}$$

In eqn (10), δW is an expression of the form

$$\delta W = \sum_{i=1}^{N_{\text{elem}}} \int_0^{L_i} [F_{v_i} \delta v_i + F_{u_i} \delta u_i] \, ds_i \tag{11}$$

and denotes the total virtual work of all other forces that are not accounted for by the strain and potential energies (i.e., damping and applied forces).

One way to proceed with the formulation consists in adjoining N_{beams} constraint equations, each of which has the form of eqn (2), to eqn (10) with N_{beams} Lagrange multipliers, and then performing the variations in that equation to obtain $2 \times N_{\text{beams}}$ equations that involve the variables $u_i(s_i, t)$ and $v_i(s_i, t)$ for each beam, as first done in Crespo da Silva and Glynn (1978a) for a single beam. The Lagrange multipliers and the variables u_i

can be eliminated from the resulting equations in order to obtain N_{beams} nonlinear integro-partial differential equations for the bending deflection $v_i(s_i, t)$ for each beam. An approximate solution based on a perturbation expansion applied to such equations would then be generated, in conjunction with a modal reduction technique based on Galerkin's method for example.

For a single beam, this "route" is convenient, but for a structure with a number of beams that make arbitrary angles $\alpha_i \neq 0$ with other adjacent beams, such route is clearly not convenient because one would end up with a large number of coupled partial differential equations to analyze the motion.

Another way to proceed with the formulation is to use a modal reduction approximation directly in eqn (10) in order to model the dynamics of the entire structure by a set of reduced-order nonlinear ordinary differential equations of motion truncated to a predetermined degree in the perturbation. This is the route that will be used here, and, as in previous work, nonlinearities up to cubic order in the perturbations will be retained in the differential equations that will be generated. To this end, the perturbed bending deflection $v_{is}(s_i, t)$ of each beam element i is approximated by an expansion in terms of an arbitrary number n of "modal functions" as given in eqn (12) below. Note that retention of the fourth degree terms in eqn (12) is necessary, since some of the coefficients of the variation δv_i that appear in eqn (10), such as the ones associated with the potential energy, will give rise to cubic order terms in the final differential equations of motion.

$$v_{is}(s_i, t) = \sum_{j=1}^{n} f_{i_j}(s_i)q_j(t) + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2} a_{2i_{jk}} q_j(t)q_k(t) + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{3i_{jkl}} q_j(t)q_k(t)q_l(t)$$
$$+ \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} a_{4i_{jklm}} q_j(t)q_k(t)q_l(t)q_m(t) + \cdots \quad (12)$$

In the above fourth-degree expansion for the bending deflection of an arbitrary element i in terms of the n modal functions f_{i_j} $(j = 1, 2, \ldots, n)$ for that element, the coefficients $a_{2i_{jk}}$, $a_{3i_{jkl}}$ and $a_{4i_{jklm}}$ are constants. The sum of the terms that depend on these quantities in eqn (12) is the $\hat{y}_i$ component of the vector $\overrightarrow{O_i O_i^*}$, shown in Fig. 1, which is viewed by element i as an equivalent "base motion" for its leftmost point O_i^*. That equivalent "base motion" is, in turn, due to the motion of another element (or elements) that may be attached to that same point. The quantities $a_{2i_{jk}}$, $a_{3i_{jkl}}$ and $a_{4i_{jklm}}$ can be determined by dividing each beam in the structure into straight elements and then spanning the structure, element by element, starting with any support whose motion is known. For an element i, connected to a support with no motion relative to O_s (see Fig. 1), $a_{2i_{jk}} = a_{3i_{jkl}} = a_{4i_{jklm}} = 0$.

By substituting the above expression for $v_{is}(s_i, t)$ into eqn (4), and integrating the resulting equation for $u_i'(s_i, t) = u_{ie}'(s_i) + u_{is}'(s_i, t)$, the following expanded expression for $u_{is}(s_i, t)$, which is needed in eqn (10), is obtained.

$$u_{is}(s_i, t) = \sum_{j=1}^{n} b_{i_j}(s_i)q_j(t) + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2} C_{i_{jk}}(s_i)q_j(t)q_k(t) + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} g_{i_{jkl}}(s_i)q_j(t)q_k(t)q_l(t)$$
$$+ \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} h_{i_{jklm}}(s)q_j(t)q_k(t)q_l(t)q_m(t) + \cdots \quad (13)$$

where

$$b_{i_j}(s_i) = b_{i_j}(0) - \int_0^{s_i} (\tan\theta_{ie}) f_{i_j}'(s_i)\, ds_i$$

$$C_{i_{jk}}(s_i) = C_{i_{jk}}(0) - \int_0^{s_i} \frac{1}{\cos^3\theta_{ie}} f_{i_j}' f_{i_k}'\, ds_i$$

$$g_{i_{jkl}}(s_i) = g_{i_{jkl}}(0) - \int_0^{s_i} \frac{\sin\theta_{ie}}{2\cos^5\theta_{ie}} f'_{i_j} f'_{i_k} f'_{i_l}\,\mathrm{d}s_i$$

$$h_{i_{jklm}}(s_i) = h_{i_{jklm}}(0) - \int_0^{s_i} \frac{(1+4\sin^2\theta_{ie})}{8\cos^7\theta_{ie}} f'_{i_j} f'_{i_k} f'_{i_l} f'_{i_m}\,\mathrm{d}s_i \tag{14}$$

The initial values $b_{i_j}(0), \ldots, h_{i_{jklm}}(0)$ for element i are determined from the $\hat{x}_i$ component of the vector $\overrightarrow{O_iO_i^*}$ shown in Fig. 1. This is also done by spanning the structure element by element, starting, as indicated above, at a support with no motion relative to point O_s.

The modal expansions for $v_i(s_i, t)$ and $u_i(s_i, t)$ given by eqns (12) and (13), and the expansion for $\theta_i(s_i, t)$ given by eqn (5) are now used in eqn (10). By performing the variations indicated in that equation, and by integrating by parts terms that involve $\dot{u}_i\delta\dot{u}_i$, $\dot{v}_i\delta\dot{v}_i$, $\dot{\theta}_i\delta\dot{\theta}_i$, and $\theta'_i\delta\theta'_i$, the following nonlinear ordinary differential equations, with nonlinearities up to the third degree in the bending deflection, eqns (15), are obtained for the n modal coordinates $q_j(t)$. The expressions for all the coefficients that appear in eqns (15) are given in the Appendix.

$$\sum_{k=1}^{n} [M_{jk}\ddot{q}_k + K_{jk}q_k] + \sum_{k=1}^{n}\sum_{l=1}^{n} [\alpha_{2_{jkl}}q_kq_l + \beta_{2_{jkl}}(\dot{q}_kq_l)^{\cdot} + \beta_{2_{kjl}}\ddot{q}_kq_l]$$

$$+ \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} [\alpha_{3_{jklm}}q_kq_lq_m + \alpha_{4_{jmkl}}(\dot{q}_kq_l)^{\cdot}q_m + \beta_{4_{jklm}}(\dot{q}_kq_lq_m)^{\cdot} + \beta_{4_{kjlm}}\ddot{q}_kq_lq_m]$$

$$+ \ddot{x}_s\left[E_{1u_j} + \sum_{k=1}^{n} E_{2u_{jk}}q_k + \sum_{k=1}^{n}\sum_{l=1}^{n} E_{3u_{jkl}}q_kq_l\right] + \ddot{y}_s\left[E_{1v_j} + \sum_{k=1}^{n} E_{2v_{jk}}q_k + \sum_{k=1}^{n}\sum_{l=1}^{n} E_{3v_{jkl}}q_kq_l\right]$$

$$+ \text{fourth and higher order polynomial nonlinearities}$$

$$= Q_{1_j} + \sum_{k=1}^{n} Q_{2_{jk}}q_k + \sum_{k=1}^{n}\sum_{l=1}^{n} Q_{3_{jkl}}q_kq_l \quad (j = 1, 2, \ldots, n) \tag{15}$$

The quantities Q_{1_j}, $Q_{2_{jk}}$ and $Q_{3_{jkl}}$ that appear in the right hand side of eqns (15) are generalized nonconservative forces associated with the n generalized coordinates q_j. They are obtained from the expression for δW given by eqn (11) and are given as shown in eqns (16) below. The terms associated with the components $\ddot{x}_s(t)$ and $\ddot{y}_s(t)$ of a prescribed "base excitation" of the entire structure produce both *direct excitation* terms (which are those associated with the coefficients E_{1u_j} and E_{1v_j}) and *parametric excitation* terms (which are the terms associated with the coefficients $E_{2u_{jk}}$, $E_{2v_{jk}}$, $E_{3u_{jkl}}$ and $E_{3v_{jkl}}$). The terms associated with the coefficients Q_{1_j}, $Q_{2_{jk}}$ and $Q_{3_{jkl}}$ also produce the same type of excitation terms when distributed forces that are explicitly dependent on time are applied to the structure.

$$Q_{1_j} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} [F_{vi}f_{i_j} + F_{ui}b_{i_j}]\,\mathrm{d}s_i$$

$$Q_{2_{jk}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} [F_{vi}a_{2i_{jk}} + F_{ui}C_{i_{jk}}]\,\mathrm{d}s_i$$

$$Q_{3_{jkl}} = 3\sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} [F_{vi}a_{3i_{jkl}} + F_{ui}g_{i_{jkl}}]\,\mathrm{d}s_i \tag{16}$$

It can be readily shown that for a single beam with constant properties, and with $v_e = 0$, the n ordinary differential equations obtained here, eqns (15), are equivalent to the ones that can be obtained from the partial differential equation (5a) in Crespo da Silva and Glynn (1978b). They are also equivalent to the ordinary differential equations that can be obtained for a beam with variable properties by using a modal reduction approximation in the partial differential equation (11b) in Crespo da Silva and Glynn (1978a).

The nonlinear differential equations developed here are in a form suited for investigations involving nonlinear motions of single or multi-beam structures using analytical techniques, such as perturbation methods, and for the design and performance analyses of linear or nonlinear control systems for those structures. Such investigations will be presented in forthcoming publications by this author.

CALCULATION METHODOLOGY

The generation of eqns (15) and the calculation of the numerical values of the coefficients of each of its terms was implemented in the form of a computer program. To calculate the coefficients whose expressions are given in the Appendix, the user specifies the location of each beam in a given undeformed structure, the properties of each beam (i.e., specific mass and stiffness), the location and values of the concentrated masses (if any) and their moments of inertia, the loads, connections between the beams, and types of supports (such as clamped, pinned-sliding, etc.).

With the above information, the equilibrium deformation and a user-specified number N of modes of linearized vibration are determined numerically using a finite element program. This author uses the "DYMORE" finite element program (Bauchau, 1993) with a four-node mixed beam element (which uses the transverse deflection and bending moment as primary degrees of freedom), but any other program that yields the nonlinear equilibrium and the modes of vibration about the equilibrium may be used. For this finite element *pre-processing* of the structure, the user also specifies the number of elements each beam is to be divided into (naturally, the structure should be divided into enough elements so that the results obtained are independent of the number of elements used).

Having the output of the finite element program, the user then chooses a subset S_n of $n \leqslant N$ modes to be used in the modal reduction formulation presented in the previous section. The program that calculates the coefficients of all the terms in the reduced-order differential equations of motion then uses the n selected modes contained in S_n as the functions $f_{i_j}(s_i)$ introduced in the previous section. By using such modes, one guarantees that the solution to the linearized counterpart to the differential equations of motion are in agreement with the solution given by the finite element analysis when the same number of basis functions are used in both solutions. The coefficients M_{jk}, K_{jk}, $\alpha_{2_{jkl}}$, etc., are then calculated numerically and displayed, by the program that calculates them, together with each term $q_1, q_2, \ldots, q_1q_2$, etc., they are associated with in each one of the n differential equations of motion.

It is well known from the theory of nonlinear oscillations (Nayfeh and Mook, 1979; Hagedorn, 1988, etc.) that modes whose natural frequencies ω_1, ω_2, etc., are related as $|N_1\omega_1+N_2\omega_2+\cdots| \approx 0$ (where $N_1, N_2, \ldots,$ are appropriate integers that depend on the order of the nonlinearities in the differential equations) are prone to interact in a nonlinear manner. Such interacting modes are said to be in internal resonance. The nonlinear differential equations of motion formulated in this paper allow the analyst to investigate such nonlinear motions. For this, the user of the work presented here would include those modes in the calculations of the coefficients of the terms in the reduced-order differential equations of motion for a particular structure under investigation, and analyze the motion as in Crespo da Silva and Zaretzky (1990) for example.

COMPARISON WITH FULL FINITE ELEMENTS AND EXPERIMENTAL RESULTS

In a recent paper presented in the literature (Bauchau and Botasso, 1994), the nonlinear response of a single beam was used in order to compare the results obtained from finite elements in space and time, and from a perturbation analysis. The results of the perturbation analysis used in that comparison were based on the nonlinear differential equations developed in Crespo da Silva and Glynn (1978b) for inextensional beams, and in Crespo da Silva (1988b) for extensional beams (where the effects of all geometric nonlinearities, including curvature, inertia and mid-plane stretching, were accounted for). The results reported in Bauchau and Botasso (1994) are shown in Figs 1, 3 and 5 in that reference. They are essentially the same as the analytical results shown in Fig. 2 in Crespo da Silva and Glynn (1978b) and in Fig. 1 in Crespo da Silva (1988b).

Although the objective in Bauchau and Botasso (1994) was to stress the versatility of a numerical procedure, the results obtained in that reference clearly demonstrated that an analytical investigation based on a reduced-order differential equation model can yield extremely accurate results. Such models are highly desirable to an analyst since they can be used to predict the response of the structure and, at the same time, allow one to analytically determine the effect of various parameters in the response and define regions, in the system's parameter space, where nonlinear effects cannot be neglected in predicting the response. In addition, if one were to design a control system (using optimal control theory, for example) taking into account the effect of the nonlinearities as the structure undergo elastic deformations, the use of explicit differential equations of motion is a necessity.

An in-depth experimental investigation of the effects of nonlinearities in both the planar and the non-planar responses of vertical clamped-free beams to external periodic excitations was presented in Zaretzky and Crespo da Silva (1994). The experiments described in that reference were performed using laboratory techniques that are not dependent on a linear response. They yielded experimental results that are in excellent agreement with the analytical predictions obtained in Crespo da Silva and Glynn (1978b) by applying Galerkin's modal reduction technique to the nonlinear partial differential equations first developed in Crespo da Silva and Glynn (1978a). As mentioned earlier, the resulting ordinary differential equations that can be obtained from Crespo da Silva and Glynn (1978b) for a single beam are equivalent to eqns (15) developed in this paper since the values for the coefficients of similar terms in both equations are the same.

EXAMPLES OF APPLICATION

Two new examples are now presented in order to illustrate the use of the formulation developed here to obtain the reduced-order nonlinear differential equations of motion of a given structure. In the examples presented below, the structures are subjected to a base excitation $\mathbf{r}_s = x_s(t)\hat{x} + y_s(t)\hat{y}$. The analysis of the response of a number of complex structures is beyond the scope of this paper and will be presented in future publications.

In the absence of external excitation, damping in the system will, of course, cause any initial motion imparted to the structure to "die out" and the system will return to its equilibrium state with the passage of time. However, if the structure is excited with a periodic function for which its Fourier representation has a frequency component Ω that is near either one of the natural frequencies of the structure or a multiple of a natural frequency ω_i (such as Ω near ω_1, $2\omega_2$, $3\omega_2$, etc.), the response is likely to exhibit a phenomenon that is known as an *external nonlinear resonance*, where the nonlinearities can play a significant role in determining the system's response. In addition, if the system also exhibits internal resonances, the response will be dominated by the modes with frequencies equal to the external and internal resonance frequencies, and modes with other frequency components will die out due to damping. As indicated earlier, a number of papers in the nonlinear oscillations literature have shown that the system's response in such cases is drastically different from that obtained from the linearized differential equations.

The examples below illustrate the process of generating the differential equations for investigating nonlinear motions in the presence of resonances. As mentioned earlier, such equations are especially suited for using a number of analytical perturbation methods such as those presented in detail in Nayfeh and Mook (1979). These methods date back from the time of the pioneer work of a number of great mathematicians, including the celebrated work of Poincaré (Poincaré, 1892).

The first example consists of a uniform horizontal cantilever of length L, of constant cross section with stiffness D Newton.meters^2, and specific mass m Kg/meter. The beam is carrying two lumped masses $M_1 = 0.48M$ at $s = L/4$, and $M_2 = 0.5M$ at $s = L/2$, where $M = mL + M_1 + M_2$ is the total mass of the structure. The moments of inertia of the lumped

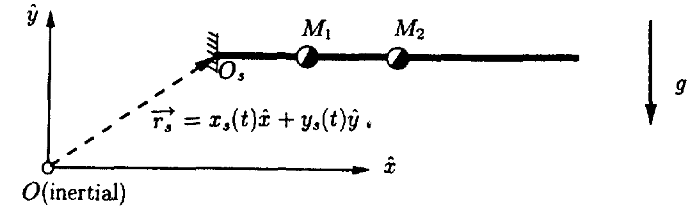

Fig. 2. A horizontal cantilever with two concentrated masses.

masses are neglected for simplicity. The structure is shown in Fig. 2. For $ML^2g/D = 4$, where $g = 9.81$ meters/s^2, a finite element pre-processing of the structure disclosed that its first three natural frequencies, nondimensionalized by $\sqrt{D/(ML^3)}$, are $\omega_1 \approx 6.63$, $\omega_2 \approx 40.9$ and $\omega_3 \approx 87.6$. For the given load, the tip equilibrium deflection of this cantilever is $v_e(s = L) = -0.261L$.

Since the natural frequency of the third mode for this structure, ω_3, is near twice the natural frequency ω_2 of the second mode, the structure is prone to exhibit nonlinear interactions involving these two modes when the external excitation has a frequency component Ω that is near the frequency of any of these modes. The nonlinear differential equations of motion involving these two modes will be generated.

To generate the reduced-order nonlinear differential equations that are suited for investigating the nonlinear interactions described above, the shape functions for the second and third modes of the beam were used in the modal reduction technique presented in this paper. With time nondimensionalized by $\sqrt{ML^3/D}$, for convenience, and the bending deflection of the beam and the base excitation displacement $\mathbf{r}_s$ nondimensionalized by L, the differential equations that were obtained are shown below. Overdots denote differentiation with respect to normalized time. Structural damping is being modeled as viscous modal damping, and is represented in these equations by the linear terms $c_1\dot{q}_1$ and $c_2\dot{q}_2$, and by the cubic terms $c_{31}\dot{q}_1^3$ and $c_{32}\dot{q}_2^3$, respectively. The cubic terms are included in order to account for possible nonlinear damping effects as determined experimentally in Zaretzky and Crespo da Silva (1994). The terms due to the external base excitation are those in the right-hand side of the differential equations. For this example, a periodic excitation with a Fourier frequency component Ω near one of the two natural frequencies being considered, or near a multiple of these natural frequencies (such as Ω near ω_3, $2\omega_2$, $3\omega_3$, etc.), will cause external nonlinear resonances where the nonlinearities in these equations will play a significant role in the system's response. An external nonlinear resonance, without internal modal interactions, will also occur when $\Omega \approx \omega_1$ for example. The nonlinear differential equation of motion for investigating such response is obtained by using the shape function of the first mode for recalculating the coefficients of the various nonlinear terms in the resulting equation. Single-mode differential equations involving the second and the third modes for this structure are obtained by simply setting $q_2 = 0$ in eqn (17) and $q_1 = 0$ in eqn (18).

$$\begin{aligned}
&\ddot{q}_1 + 1672q_1 + c_1\dot{q}_1 + c_{31}\dot{q}_1^3 + 0.14q_2 + 3.04[(\dot{q}_1q_1)^{\cdot} + \ddot{q}_1q_1] + 2.77(\dot{q}_1q_2)^{\cdot} \\
&\quad + 332q_1(\dot{q}_1q_1)^{\cdot} + 48.1[(q_1q_2\dot{q}_1)^{\cdot} + q_1(\dot{q}_1q_2)^{\cdot}] + 270(\dot{q}_1q_2^2)^{\cdot} \\
&\quad + \ddot{q}_2(1.47q_1 + 10.13q_2 + 41.41q_1^2 + 699q_1q_2 + 585q_2^2) \\
&\quad + \dot{q}_2^2(6.09 + 451q_1 + 607q_2) + q_1^2(2215 + 108{,}035q_1 + 493{,}540q_2) \\
&\quad + q_2^2(33{,}072 + 2{,}299{,}382q_1 + 2{,}451{,}534q_2) + 14{,}151q_1q_2 \\
&= \ddot{x}_s[0.0439 + 14.83q_1 - 3.54q_2 - 22.28q_1^2 - 19.94q_1q_2 - 26.06q_2^2] - 0.477\ddot{y}_s
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\ddot{q}_2 + 7676q_2 + c_2\dot{q}_2 + c_{32}\dot{q}_2^3 + 0.14q_1 + 8.49[(\dot{q}_2q_2)^{\cdot} + \ddot{q}_2q_2] + 8.06(\dot{q}_2q_1)^{\cdot} \\
&\quad + 1672q_2(\dot{q}_2q_2)^{\cdot} + 562[(q_2q_1\dot{q}_2)^{\cdot} + q_2(\dot{q}_2q_1)^{\cdot}] + 248(\dot{q}_2q_1^2)^{\cdot}
\end{aligned}$$

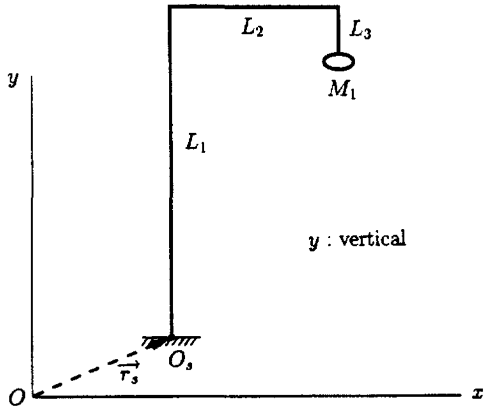

Fig. 3. A three-beam structure with a concentrated mass.

$$
\begin{aligned}
&+\ddot{q}_1(1.47q_1+10.13q_2+41.41q_1^2+699q_1q_2+585q_2^2)\\
&+\dot{q}_1^2(0.087+34.8q_1+429q_2)+q_1^2(7075+164{,}513q_1+2{,}299{,}382q_2)\\
&+q_2^2(104{,}666+7{,}354{,}601q_1+8{,}177{,}184q_2)+66{,}143q_1q_2\\
&=-\ddot{x}_s[0.05+3.54q_1-8.96q_2+9.97q_1^2+52q_1q_2+39q_2^2]+0.102\ddot{y}_s \qquad (18)
\end{aligned}
$$

As indicated by eqns (17) and (18), the component of the external base excitation that is perpendicular to the undeformed direction of the structure shown in Fig. 2 only gives rise to a direct excitation term (which are the terms in $\ddot{y}_s$), while the $\ddot{x}_s$ component of the excitation produces both direct and parametric excitation terms. Some of the parametric excitation terms will also produce a number of nonlinear resonances when the external excitation is periodic.

It is interesting to notice that the differential equations of motion of a number of other systems have the same general form of the above equations. Such is the case, for example, of the model analyzed in Ibrahim *et al.* (1988) and in Ibrahim and Li (1988) for the response of an elevated water tower containing a liquid with a free surface, subjected to ground motion.

The second example presented here consists of a three-beam vertical structure shown in Fig. 3. Each beam in the structure is made of the same material and has the same mass density m Kg/meter, and bending stiffness D Newton.meter^2. The connection between the beams is a weld, and the beam of length L_1 is vertical and is clamped to a support that is subjected to a given base displacement $\mathbf{r}_s(t) = x_s(t)\hat{x}+y_s(t)\hat{y}$. A small body of mass M_1, and negligible moment of inertia, is attached to the end of the structure. The total length of the structure is $L = L_1+L_2+L_3$ and its total mass is $M = mL+M_1$. For $L_1/L = 0.6$, $L_2/L = 0.3$, $L_3/L = 0.1$, $M_1/M = 0.1$, and $ML^2g/D = 1$, the first three-nondimensional natural frequencies for this structure are $\omega_1 \approx 4.4$, $\omega_2 \approx 12.7$ and $\omega_3 \approx 53.6$, with the time nondimensionalization factor chosen as $\sqrt{ML^3/D}$ sec. Since $\omega_2 \approx 3\omega_1$, nonlinear interactions involving the first and second modes, and caused by cubic nonlinearities in the system, are prone to occur in this structure.

The reduced-order nonlinear differential equations of motion for this structure, obtained by using the first and the second natural modes in the modeling presented in this paper, are given below. In these equations, overdots denote differentiation with respect to the normalized time used for this structure. The bending deflection of the structure and the base displacement $\mathbf{r}_s$ are also nondimensionalized by the total length L of the structure.

$$
\begin{aligned}
&\ddot{q}_1 + 19.17q_1 + c_1\dot{q}_1 + c_{31}\dot{q}_1^3 - 0.26q_2 - 0.145[(\dot{q}_1 q_1)^{\cdot} + \ddot{q}_1 q_1] - 4.091(\dot{q}_1 q_2)^{\cdot} \\
&\quad + 6.524q_1(\dot{q}_1 q_1)^{\cdot} + 1.338[(q_1 q_2 \dot{q}_1)^{\cdot} + q_1(\dot{q}_1 q_2)^{\cdot}] + 5.596(\dot{q}_1 q_2^2)^{\cdot} \\
&\quad - \ddot{q}_2(0.012q_1 + 2.07q_2 - 2.79q_1^2 - 14.74q_1 q_2 - 10.67q_2^2) \\
&\quad - \dot{q}_2^2(2.74 - 5.47q_1 - 7.41q_2) + q_1^2(7.136 + 96.3q_1 + 277q_2) \\
&\quad + q_2^2(46.43 + 1580q_1 + 1120q_2) + 23.65q_1 q_2 \\
&= -\ddot{x}_s[0.725 - 1.035q_1 - 1.613q_2 + 0.466q_1^2 + 1.472q_1 q_2 + 1.207q_2^2] \\
&\quad + \ddot{y}_s[0.35 + 1.715q_1 - 0.784q_2 - 0.346q_1^2 + 0.257q_1 q_2 + 0.173q_2^2]
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&\ddot{q}_2 + 161.1q_2 + c_2\dot{q}_2 + c_{32}\dot{q}_2^3 - 0.26q_1 + 1.86[(\dot{q}_2 q_2)^{\cdot} + \ddot{q}_2 q_2] + 1.34(\dot{q}_2 q_1)^{\cdot} \\
&\quad + 23.52q_2(\dot{q}_2 q_2)^{\cdot} + 13.9[(q_2 q_1 \dot{q}_2)^{\cdot} + q_2(\dot{q}_2 q_1)^{\cdot}] + 9.27(\dot{q}_2 q_1^2)^{\cdot} \\
&\quad - \ddot{q}_1(0.012q_1 + 2.07q_2 - 2.79q_1^2 - 14.74q_1 q_2 - 10.67q_2^2) \\
&\quad + \dot{q}_1^2(2.034 + 4.25q_1 + 9.15q_2) + q_1^2(11.83 + 92.3q_1 + 1580q_2) \\
&\quad + q_2^2(103.7 + 3360q_1 + 2834q_2) + 92.9q_1 q_2 \\
&= \ddot{x}_s[0.513 + 1.613q_1 + 2.619q_2 - 0.736q_1^2 - 2.415q_1 q_2 - 2.045q_2^2] \\
&\quad + \ddot{y}_s[0.475 - 0.784q_1 + 0.298q_2 + 0.128q_1^2 + 0.347q_1 q_2 + 0.636q_2^2]
\end{aligned} \tag{20}
$$

Unless $L_2 = 0$, both the $\ddot{x}_s$ and the $\ddot{y}_s$ components of the external base excitation produce parametric excitation terms in this structure. This occurs because the undeformed structure is not a straight line when $L_2 \neq 0$.

The natural frequencies of the systems shown in Figs 2 and 3 depend, of course, on several factors such as the location of the concentrated masses, on the number of such masses, and on the values of the masses. The differential equations of motion for the examples presented above exhibit internal nonlinear resonances. They also exhibit external resonances when the base excitation is periodic and the frequency, Ω, of a component of its Fourier series is near one of the natural frequencies of the system, or near a multiple of a natural frequency ω_i. These external resonances, which occur when $\Omega \approx 2\omega_i$, $\Omega \approx 3\omega_i$, etc., may produce phenomena that are characterized by the fact that the actual motion exhibited by the system bears no resemblance to the motion described by the linearized differential equations.

CONCLUDING REMARKS

The work presented in this paper constitutes a hybrid formulation that combines the advantages of both analytical and numerical methods to analyze a class of multi-beam structures. Structures for which each beam in the structure behaves as inextensional were considered in this paper.

A finite element pre-processing enables the analyst to deal with large structures and provides information that is used for generating the coefficients of all the terms that appear in the explicit differential equations of motion. The equations developed here allow for each beam in the structure to have arbitrary property variations along its span. Such equations model the nonlinear dynamics of the entire structure and allow the analyst to perform investigations that would not be possible to conduct with numerical simulations only. These investigations include the prediction and the analysis of nonlinear phenomena exhibited by the structure, evaluation of the effect of structural nonlinearities in the performance of control systems that are designed on the basis of linear theory only, and the design of optimal control systems taking into account the effect of the nonlinearities in the structure due to finite deformation of any of its members.

Nonlinear motions can be drastically different from those predicted by using linearized differential equations, and analytical investigations of such motions are crucial for understanding them. This author views the work presented here as a first step in the analytical modeling of the dynamics of multi-beam structures. Extension of this work to more general class of structures including extensional members, and, at the end of the complexity spectrum, to three-dimensional structures that can undergo flexure in any direction in space, torsion and extension, is highly desirable.

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APPENDIX

The expressions for all the coefficients in the differential equations eqns (15), are given below. To calculate them, each beam in the structure is divided into a desired number of elements whose length is L_i; N_{elem} is the total number of these elements in the structure.

$$M_{jk} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i(f_{i_j} f_{i_k} + b_{i_j} b_{i_k})\,\mathrm{d}s_i$$

$$+ \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[f_{i_j} f_{i_k} + b_{i_j} b_{i_k}] + J_{i_{\text{conc}}}(A_{1i} f'_{i_j})(A_{1i} f'_{i_k})\}_{s_i = s_{i_{\text{conc}}}}$$

$$K_{jk} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} \{D_i[(A_{1i} f'_{i_j})'(A_{1i} f'_{i_k})' + 2\theta'_e(A_{2i} f'_{i_j} f'_{i_k})']$$

$$+ m_i \boldsymbol{g}[C_{i_{jk}} \sin\alpha_i + \alpha_{2i_{jk}} \cos\alpha_i]\}\,\mathrm{d}s_i$$

$$+ \boldsymbol{g} \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[C_{i_{jk}} \sin\alpha_i + a_{2i_{jk}} \cos\alpha_i]\}_{s_i = s_{i_{\text{conc}}}}$$

$$\alpha_{2_{jkl}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} \{D_i[(A_{1i} f'_{i_j})'(A_{2i} f'_{i_k} f'_{i_l})' + 2(A_{1i} f'_{i_l})'(A_{2i} f'_{i_j} f'_{i_k})'$$

$$+ 3\theta'_{ie}(A_{3i} f'_{i_j} f'_{i_k} f'_{i_l})'] + 3 m_i \boldsymbol{g}[\boldsymbol{g}_{i_{jkl}} \sin\alpha_i + a_{3i_{jkl}} \cos\alpha_i]\}\,\mathrm{d}s_i$$

$$+ 3\boldsymbol{g} \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[\boldsymbol{g}_{i_{jkl}} \sin\alpha_i + a_{3i_{jkl}} \cos\alpha_i]\}_{s_i = s_{i_{\text{conc}}}}$$

$$\beta_{2_{jkl}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i[b_{i_j} C_{i_{kl}} + f_{i_j} a_{2i_{kl}}]\,\mathrm{d}s_i$$

$$+ \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[b_{i_j} C_{i_{kl}} + f_{i_j} a_{2i_{kl}}] + 2J_{i_{\text{conc}}}(A_{1i} f'_{i_j})(A_{2i} f'_{i_k} f'_{i_l})\}_{s_i = s_{i_{\text{conc}}}}$$

$$\alpha_{3_{jklm}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} \{D_i[(A_{1i} f'_{i_j})'(A_{3i} f'_{i_k} f'_{i_l} f'_{i_m})' + 2(A_{2i} f'_{i_j} f'_{i_k})'(A_{2i} f'_{i_l} f'_{i_m})'$$

$$+ 3(A_{3i} f'_{i_j} f'_{i_k} f'_{i_l})'(A_{1i} f'_{i_m})' + 4\theta'_{ie}(A_{4i} f'_{i_j} f'_{i_k} f'_{i_l} f'_{i_m})']$$

$$+ 4 m_i \boldsymbol{g}[\boldsymbol{g}_{i_{jklm}} \sin\alpha_i + a_{4i_{jklm}} \cos\alpha_i]\}\,\mathrm{d}s_i$$

$$+ 4\boldsymbol{g} \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[h_{i_{jklm}} \sin\alpha_i + a_{4i_{jklm}} \cos\alpha_i]\}_{s_i = s_{i_{\text{conc}}}}$$

$$\alpha_{4_{jklm}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i[C_{i_{jk}} C_{i_{lm}} + a_{2i_{jk}} a_{2i_{lm}}]\,\mathrm{d}s_i$$

$$+ \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[C_{i_{jk}} C_{i_{lm}} + a_{2i_{jk}} a_{2i_{lm}}] + 4J_{i_{\text{conc}}}(A_{2i} f'_{i_j} f'_{i_m})(A_{2i} f'_{i_k} f'_{i_l})\}_{s_i = s_{i_{\text{conc}}}}$$

$$\beta_{4_{jklm}} = 3 \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i[b_{i_j} \boldsymbol{g}_{i_{klm}} + f_{i_j} a_{3i_{klm}}]\,\mathrm{d}s_i$$

$$+ 3 \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}}[b_{i_j} \boldsymbol{g}_{i_{klm}} + f_{i_j} a_{3i_{klm}}] + J_{i_{\text{conc}}}(A_{1i} f'_{i_j})(A_{3i} f'_{i_k} f'_{i_l} f'_{i_m})\}_{s_i = s_{i_{\text{conc}}}}$$

$$E_{1u_j} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i b_{xi_j} \, \mathrm{d}s_i + \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}} b_{xi_j}\}_{s_i = s_{i_{\text{conc}}}}$$

$$E_{1v_j} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i b_{yi_j} \, \mathrm{d}s_i + \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}} b_{yi_j}\}_{s_i = s_{i_{\text{conc}}}}$$

$$E_{2u_{jk}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i C_{xi_{jk}} \, \mathrm{d}s_i + \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}} C_{xi_{jk}}\}_{s_i = s_{i_{\text{conc}}}}$$

$$E_{2v_{jk}} = \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i C_{yi_{jk}} \, \mathrm{d}s_i + \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}} C_{yi_{jk}}\}_{s_i = s_{i_{\text{conc}}}}$$

$$E_{3u_{jkl}} = 3 \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i g_{xi_{jkl}} \, \mathrm{d}s_i + 3 \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}} g_{xi_{jkl}}\}_{s_i = s_{i_{\text{conc}}}}$$

$$E_{3v_{jkl}} = 3 \sum_{i=1}^{N_{\text{elem}}} \int_{s_i=0}^{L_i} m_i g_{yi_{jkl}} \, \mathrm{d}s_i + 3 \sum_{i=1}^{N_{\text{conc}}} \{M_{i_{\text{conc}}} g_{yi_{jkl}}\}_{s_i = s_{i_{\text{conc}}}} \tag{A1}$$

where

$$b_{xi_j} = b_{i_j} \cos\alpha_i - f_{i_j} \sin\alpha_i$$

$$b_{yi_j} = b_{i_j} \sin\alpha_i + f_{i_j} \cos\alpha_i$$

$$C_{xi_{jk}} = C_{i_{jk}} \cos\alpha_i - a_{2i_{jk}} \sin\alpha_i$$

$$C_{yi_{jk}} = C_{i_{jk}} \sin\alpha_i + a_{2i_{jk}} \cos\alpha_i$$

$$g_{xi_{jkl}} = g_{i_{jkl}} \cos\alpha_i - a_{3i_{jkl}} \sin\alpha_i$$

$$g_{yi_{jkl}} = g_{i_{jkl}} \sin\alpha_i + a_{3i_{jkl}} \cos\alpha_i \tag{A2}$$